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Hysteresis, Discrete Memory and Nonlinear Wave Propagation in Rock: A New Paradigm

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Abstract

The structural elements in a rock are characterized by their density in Preisach-Mayergoyz space (P-M space). This density is found for a Berea sandstone from stress-strain data and used to study the response of the sandstone to elaborate pressure protocols. Hysteresis with discrete memory, in agreement with experiment, is found. The relationship between strain, quasistatic modulus, and dynamic modulus is established. Nonlinear wave propagation, the production of copious harmonics and nonlinear attenuation are demonstrated. P-M space is shown to be the central construct in a new paradigm for the description of the elastic behavior of consolidated materials.

91.60.Ba, 91.60.Fe, 91.60.Lj, 83.80.Nb

Rocks at low pressure, $P \leq 1000$ atm, have remarkable elastic properties. Their stress-strain equation of state is hysteretic, possessing discrete memory [1]. Their third order elastic constants, measures of their nonlinearity, are typically three to four orders of magnitude greater than those of materials such as Al and SiO_2 [2]. Amplitude dependent attenuation is commonly encountered [3]. Rocks are an example of a consolidated material. Such materials are not properly described by the traditional theory of nonlinear elasticity [4,5]. The purpose of this paper is to illustrate some quantitative features of a new paradigm treating the elasticity of consolidated materials, and to describe results for elastic wave propagation from use of this paradigm.

The unusual elastic behavior of rock is due primarily to the the mesoscopic structural features in rock, e.g., grain contacts, cracks, voids, ... We want to discuss a theoretical framework for the description of the macroscopic nonlinear elastic response of a material containing many mesoscopic structural features. The centerpiece of the framework is Preisach-Mayergoyz space (P-M space) [6,7], a density space for the mesoscopic structural features and their elastic properties. McCall and Guyer [8] showed how to use a known P-M space density to understand and describe hysteresis, discrete memory, and many other elastic properties of rock. They argued that a suitable experiment would provide the means to determine the content of P-M space. Here we illustrate the journey from a suitable experiment to the contents of P-M space and to predictive power for the elastic properties of the rock. The elements of this theoretical framework are markedly different from those of the traditional analytic theories of nonlinear elasticity [4,5] and constitute a new paradigm for the elasticity of consolidated materials.

The fundamental premise underlying our description of a rock is that macroscopic elastic behavior is due primarily to a large number of mesoscopic structural features. We call these features hysteretic mesoscopic units (HMU). To see the consequences of this premise we characterize the individual HMU and evaluate the influence of an ensemble of HMU on macroscopic behavior. We describe the HMU with a small number of parameters. For example, the individual HMU may be modeled by mechanical features whose equilibrium lengths switch hysteretically between two configurations, open and closed, at pressures P_o and P_c respectively (Fig. 1). [We use the language of cracks for convenience while not being committed to a particular model of the HMU.] In the closed (open) configuration the unit has equilibrium length ℓ_c (ℓ_o). The pressures (P_c, P_o) are pressures across the unit, a simple approximation to the local stress field across a structural feature. The structural features that we model in this fashion have complex local stress fields and complex responses to pressure. Since, however, we are describing a system with many such units, we believe that these details are extraneous compared to the essential features captured by a set such as $(\ell_c, \ell_o, P_c, P_o)$.

To follow the behavior of a large number of HMU we use the Preisach-Mayergoyz picture and describe the elastic state of the system as a trajectory in P-M space [9]. This paradigm has been used successfully to describe a wide variety of hysteretic systems. In our context P-M space is the space in which we locate the pressure pairs (P_c, P_o) corresponding to the set of HMU in the rock. The elastic state of the system is the consequence of being brought to pressure P by a prescribed pressure protocol. This protocol leads to an elastic state trajectory E crossing P-M space [9,10]. The stress-strain and modulus-stress relationships are calculated from E and the density $\rho(P_c, P_o)$ of HMU in P-M space. Thus these equations of state are functionals of the elastic state [8].

For illustrative purposes we assume that all of the HMU share the same two values of ℓ and are configured as a cubic lattice [8]. At uniaxial pressure P and elastic state E, the length of the system L is given by

$$L(E) = \ell_o N_T + (\ell_c - \ell_o) N(E), \tag{1}$$

where $N(E)$ is the number of closed HMU in elastic state E and $\ell_o N_T$ is the length of the system at zero pressure. The strain, defined with respect to the initial state of the system, $N(E) = 0$, is given by

$$\epsilon(E) = \frac{L(E) - L(0)}{L(0)} = -\alpha n(E), \tag{2}$$

where $\alpha = (\ell_o - \ell_c)/\ell_o$, $\ell_o > \ell_c$, and $n(E) = N(E)/N_T$. A stress-strain equation of state having hysteresis loops with discrete memory follows immediately from this equation [8,9]. The elastic modulus $M(E)$ is given by

$$M(E)^{-1} = -\frac{\partial \epsilon}{\partial P} = \alpha \frac{\partial n(E)}{\partial P}. \tag{3}$$

Thus we see that $M(E)^{-1}$ has a close connection to $n(E)$ and to the density $\rho(P_c, P_o)$. Modulus-stress data yield $\partial n(E)/\partial P$, simple integrals over the P-M space density. We can use data $M(E)^{-1}$ data to learn the P-M space density.

In Fig. 2 we show the inverse of the quasistatic elastic modulus as a function of stress for a room-dry Berea sandstone. This inverse modulus was found by differentiating stress-strain data generated in uniaxial compressional tests using the pressure protocol shown in the inset [11]. If we coarse grain P-M space on pressure scale ΔP into $K(K+1)/2$ cells and define $p(i,j) = \rho(P_i, P_j)(\Delta P)^2$ as the contents of cell (i,j), $i,j = 1 \ldots K$, then when the pressure is increased in steps of size ΔP the elastic modulus is given by

$$M(P_i)^{-1} = \alpha \sum_{j=1}^{i} p(i,j), \tag{4a}$$

and when the pressure is decreased in steps of size ΔP the modulus is given by

$$M(P_j)^{-1} = \alpha \sum_{i=j}^{K} p(i,j). \tag{4b}$$

To find $p(i,j)$ using Eq. (4), we coarse grained the largest loop in Fig. 2 using $\Delta P =$ 0.2 MPa, such that $K = 65$. Thus we had 130 constraints to fix $K(K+1)/2 = 4290$ values of $p(i,j)$. To invert for $p(i,j)$ we used a simulated annealing scheme with quadratic smoothing [12]. The results of this analysis are shown in Fig. 3 on a gray scale in which the darkest squares correspond to the highest density $p(i,j)$. Several observations are in order.

1. Approximately 50% of the HMU are off the diagonal, i.e., hysteretic. The HMU on the diagonal (nonhysteretic) constitute an increasing percentage of a decreasing number of active units as the ambient stress is increased. The off-diagonal HMU follow a similar pattern, producing the strain hardening exhibited in the data as well as the decrease in degree of hysteresis with increasing stress.

2. The density $p(i,j)$ is largest in the low pressure corner. Thus as the pressure is increased the strain changes most rapidly with pressure at low pressure. The HMU along the P_c axis are responsible for a low pressure deformation that does not relax until the sample is unloaded.

3. We have calculated the quasistatic modulus-pressure equation of state using Eq. (1), Eq. (3), and $p(i,j)$ from Fig. 3 for the complete pressure protocol in Fig. 2. We find the modulus vs pressure shown by the solid curve in Fig. 4. Comparison of Figs. 2 and 4 confirms the reasonableness of $p(i,j)$ in Fig. 3.

4. At low frequency, a sinusoidal wave of pressure amplitude δP carries the material at (x,t) through the same series of elastic states as a sinusoidal quasistatic pressure protocol. Figure 2 shows the evolution of the elastic modulus with δP as the strain is reduced from 10^{-3} to 10^{-4}. As the strain amplitude is made smaller and smaller, the amplitude of the hysteresis loop becomes correspondingly smaller while the qualitative properties of the loop remain unchanged. We estimate the dynamic modulus by neglecting frequency effects and using the smallest quasistatic pressure cycle that we can resolve [13–15], $\Delta P = 0.2$ MPa. We find the result shown as open circles in Fig. 4. The dynamic modulus is larger than the quasistatic modulus except at the turning points where they coincide. The dynamic modulus is related to the behavior of $p(i,j)$ near the diagonal in P-M space, whereas the quasistatic modulus is related to $p(i,j)$ throughout P-M space [Eq. (4)]. Thus the difference between the quasistatic and dynamic moduli is intimately related to the presence of hysteresis.

To develop a theoretical description of wave propagation we exploit observation (4). Take the rock to be in elastic state E_0. When there is a pressure disturbance δP in the rock we take the elastic modulus at (x,t) to be

$$M(x,t;E_0) = M_0\{1+\kappa[\delta P(x,t)]\}, \tag{5}$$

where M_0 is the elastic modulus at the ambient elastic state E_0 of the rock and κ, a functional of the pressure disturbance at (x,t), is found from Eqs. (1) – (3) and $\rho(i,j)$ in Fig. 3. The time or space average of κ is zero. To describe one dimensional wave propagation we take the equation of motion for the displacement field to be

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\partial}{\partial x}\left[M_0(1+\kappa[\delta P])\frac{\partial u}{\partial x}\right]. \tag{6}$$

Suppose a single frequency disturbance propagates through the system with displacement amplitude $u_0(x,t) = U\sin\tau$, where $\tau = k_0 x - \omega_0 t$. The first order modification of u_0 due to nonlinear elasticity is given in the frequency domain by [16]

$$\begin{aligned} u_1(x,\omega) = &\int dx' \int \frac{d\omega'}{2\pi} g(x,x',\omega) \\ &\cdot \frac{\partial}{\partial x'}\left[\kappa[\delta P_0(x',\omega')]\frac{\partial u_0(x',\omega-\omega')}{\partial x'}\right], \end{aligned} \tag{7}$$

where $\delta P_0 \propto \cos\tau$ is the pressure disturbance due to u_0 and $g(x, x', \omega)$ is the Green function for the harmonic problem. The modulus $\kappa[\delta P_0]$ can be written as a sum over Fourier components in and out of phase with the pressure at frequencies $n\omega$, where $n = 1, 2, 3\ldots$, i.e.,

$$\kappa[\delta P_0] = \sum_{n=1}^{\infty} \left[a_n \cos n\tau + b_n \sin n\tau \right]. \tag{8}$$

The out of phase component of $\kappa[\delta P_0]$ is due to hysteresis, i.e., the off diagonal density in P-M space.

To find the first order nonlinear component of the displacement u_1 we insert the Green function for the harmonic problem in an infinite homogeneous space into Eq. (8) and find

$$u_1 = -\frac{k_0 U x}{2} \cos\tau \sum_{n=1}^{\infty} c_n \cos(n\tau - \phi_n), \tag{9}$$

where $c_n = \sqrt{a_n^2 + b_n^2}$ and $\tan\phi_n = b_n/a_n$. Notable features of this result are:

1. proportionality to propagation distance x;
2. proportionality to U^2 (the amplitudes a_n and b_n are proportional to δP_0 which in turn is proportional to U);
3. proportionality to traditional measures of nonlinearity; for example, the amplitude a_1 is the β coefficient of traditional analytic treatments;
4. a rich harmonic structure with amplitude proportional to U^2 at 3ω, 4ω, ... These terms are a manifestation of the discontinuous character of the response of the modulus to pressure. In the traditional analytic treatment, terms of this type are proportional to U^3 and higher powers of U.

Finally, we find the net work done by the effective stress σ as the wave propagates by calculating

$$\Delta E = \frac{1}{\rho} \oint \sigma d\epsilon \tag{10}$$

for one period of the initial disturbance, where

$$\sigma = M_0 \{1 + \kappa[\delta P(x,t)]\} \frac{\partial u}{\partial x}. \tag{11}$$

Using the lowest order treatment of the displacement field, we find [17]

$$\frac{1}{Q} - \frac{1}{Q_0} \propto b_2, \tag{12}$$

where Q_0 is due to linear attenuation mechanisms. The out-of-phase (hysteretic) component of the nonlinear elasticity is the source of amplitude dependent attenuation.

In this paper we have argued that the macroscopic elasticity of rock is due primarily to a large number of hysteretic mesoscopic units (HMU). We introduced P-M space to follow the behavior of a collection of HMU. From a stress-strain data set on a Berea sandstone we found $\rho(i,j)$, the density of HMU in P-M space. This density lets us describe the response of the rock to a complex pressure protocol and to examine the relationship between the quasistatic and the dynamic modulus. It also provides essential input for the description of wave propagation. We found copious harmonics with amplitude proportional to the square of the displacement field and nonlinear attenuation. Experimental measurements on rock show the same properties: hysteresis, discrete memory, copious production of higher harmonics [18], and nonlinear attenuation [3]. None of these properties has an easy explanation, either qualitatively or quantitatively, using the traditional analytic models of nonlinear elasticity [4,5]. Finding all of these properties in a single model is gratifying.

In addition quantitative use of the P-M space density yields a quantitative description of the bent tuning-fork behavior seen in resonant bar experiments [19]. We believe the P-M space density and its use as illustrated in this paper constitute a new paradigm for the treatment of the elastic properties of consolidated materials.

Several limitations to the demonstration we have made in this paper point to the direction of future work. The P-M space model must be developed further to describe interacting systems of hysteretic units as is called for by hysteresis in the elastic modulus or elastic avalanches. A data set as simple as a single stress-strain loop can not expose the nature of the structural elements at work in the rock. The simplicity of the P-M model lets one contemplate a series of interactive pressure protocols, in which porosity and saturation are simultaneously monitored, designed for the purpose of learning nature of the structural elements.

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FIGURES

FIG. 1. Rudimentary Elastic Unit. The elastic properties of the macroscopic system are due to an ensemble of hysteretic mesoscopic units (HMU). A unit is modeled as having an equilibrium length which goes between two states hysteretically.

FIG. 2. Modulus vs Stress I. The inverse of the Young's modulus is plotted as a function of stress for uniaxial compression of a Berea sandstone using the load history shown in the inset. The apparatus and details of how the modulus is found from stress-strain data are described in Ref. [11].

FIG. 3. P-M Space. The P-M space appropriate to the modulus-stress equation of state data in Fig. 2 is shown as a gray scale plot. About 50% of the density is on the diagonal.

FIG. 4. Modulus vs Stress II. The inverse Young's modulus, calculated using the P-M space density in Fig. 3, is plotted as a function of pressure. The load history used to construct this plot is shown in the inset of Fig. 2. The solid curve is the quasistatic modulus. The open circles are the dynamic modulus.